Determinant

Most said determinants are square schemes.

Determinant is a function depending on *n* that associates a scalar , det(*A*), to every $n \times n$ square matrix *A*.

Must be careful how marked determinants, because it is similar mark, as matrices.

By agreement, matrix is marking with square brackets, for example $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$.

Determinant is marking with $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ or $\det(A) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Finding determinants

2×2 (2 rows and 2 columns)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Multiply the elements of the so-called main diagonal minus multiplied elements of the supporting diagonal.

Example:

$$\begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} = 3 \circ 7 - 4 \circ 5 = 21-20 = 1 \qquad \begin{vmatrix} -1 & 3 \\ -5 & 12 \end{vmatrix} = (-1) \circ 12 - (-5) \circ 3 = -12+15 = 3$$

3×3 (3 rows and 3 columns)

Determinants 3×3 ,we can develop in any row or column. First of all, to every element assign sign + or -

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Just to remind you: rows are $\longrightarrow$, and column $\downarrow$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ = If you want to develop for example, by the first row =

$$=+a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$ =Or if we do development by the second column:

$$= -b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

The best is, of course, to develop by that column or the row where where we have the most zeros !

Example: Calculate the value of determinant $\begin{vmatrix} 5 & 3 & 1 \\ 1 & 7 & 0 \\ 2 & 3 & 2 \end{vmatrix}$ =?

Solution:

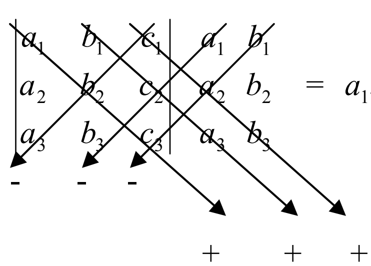
$\begin{vmatrix} 5 & 3 & 1 \\ 1 & 7 & 0 \\ 2 & 3 & 2 \end{vmatrix}$ = First, above each number write the sign: $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$, or if is easier to you ,just above the numbers in the column or row , on which you decided to expand determinant.

We have decided on the second row because it has a zero .

$$\begin{vmatrix} 5 & 3 & 1 \\ 1 & 7 & 0 \\ 2 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 5 & 3 & 1 \\ \overset{-}{1} & \overset{+}{7} & \overset{-}{0} \\ 2 & 3 & 2 \end{vmatrix} = -1\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} + 7\begin{vmatrix} 5 & 1 \\ 2 & 2 \end{vmatrix} - 0\begin{vmatrix} 5 & 3 \\ 2 & 3 \end{vmatrix}$$ =-1(3∘2 - 1∘3)+7(5∘2-2∘1)= -3 +56 = 53

Rule of Sarrus

The sum of the products of three diagonal north-west to south-east lines of matrix elements, minus the sum of the products of three diagonal south-west to north-east lines of elements when the copies of the first two columns of the matrix are written beside it as below:



$$= a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - b_1a_2c_3 - a_1c_2b_3 - c_1b_2a_3$$

Example: Calculate the value of determinant $\begin{vmatrix} 5 & 3 & 1 \\ 1 & 7 & 0 \\ 2 & 3 & 2 \end{vmatrix} = ?$

$$\begin{vmatrix} 5 & 3 & 1 \\ 1 & 7 & 0 \\ 2 & 3 & 2 \end{vmatrix} \begin{matrix} 5 & 3 \\ 1 & 7 \\ 2 & 3 \end{matrix} = 5\circ 7\circ 2 + 3\circ 0\circ 2 + 1\circ 1\circ 3 - 3\circ 1\circ 2 - 5\circ 0\circ 3 - 1\circ 7\circ 2 = 70 + 0 + 3 - 6 - 0 - 14 = 53$$

So, both ways, we get the same result, and you can select what is easier.

4×4 **(4 rows and 4 columns)**

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \text{We can develop it by any row or column! Add signs : } \begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

We will, for example, to develop determined by **the first column**:

$$\begin{vmatrix} \overset{+}{a_1} & b_1 & c_1 & d_1 \\ \overset{-}{a_2} & b_2 & c_2 & d_2 \\ \overset{+}{a_3} & b_3 & c_3 & d_3 \\ \overset{-}{a_4} & b_4 & c_4 & d_4 \end{vmatrix} = +a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

Of course, now we have to develop each of the four determinants with 3 rows and 3 columns ....

Agree that this is not easy.

DETERMINANT CHARACTERISTICS will help us in solving problems!

DETERMINANT CHARACTERISTICS

1. **Determinant changes sign if the two rows of columns change their places.**

2. **Value of determinants does not change if all rows and columns change their places.**

3. $$\mathbf{k}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1k & b_1k & c_1k \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1k & b_1 & c_1 \\ a_2k & b_2 & c_2 \\ a_3k & b_3 & c_3 \end{vmatrix} = \textbf{etc.}$$

$$\begin{vmatrix} a_1 & mb_1 & c_1 \\ a_2 & mb_2 & c_2 \\ a_3 & mb_3 & c_3 \end{vmatrix} = \mathrm{m}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

4. $$\begin{vmatrix} a_1 & b_1+m_1 & c_1 \\ a_2 & b_2+m_2 & c_2 \\ a_3 & b_3+m_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & m_1 & c_1 \\ a_2 & m_2 & c_2 \\ a_3 & m_3 & c_3 \end{vmatrix}$$

5. **If all elements of a row (column) equal to zero, the value of determinant is zero.**

$$\begin{vmatrix} 3 & 0 & 55 \\ 1 & 0 & 4 \\ 0 & 0 & 12 \end{vmatrix} = 0 \qquad \text{or} \qquad \begin{vmatrix} 0 & 0 & 0 & 0 \\ 77 & 68 & 34 & -80 \\ 8 & 5 & 7 & 4 \\ 4 & 5 & 9 & 8 \end{vmatrix} = 0$$

6. **If elements of the two rows (or columns) have the same value, of determinant is zero, again.**

Example:

$$\begin{vmatrix} 12 & 7 & 3 \\ -9 & 4 & 6 \\ 12 & 7 & 3 \end{vmatrix} = 0$$

7. **If the two species (columns) proportional to one another, the value determinants is zero .**

Example:

$$\xrightarrow{*5} \quad \begin{vmatrix} 2 & 3 & 4 \\ -9 & 5 & 56 \\ 10 & 15 & 20 \end{vmatrix} = 0$$

8. **Value of determinant remains unchanged if the elements of a row (column) is added to the elements of some other row (column). [we can multiply that row with the same number if we need that!]**

The eight properties will help us to solve determinants 4×4**,** 5×5 **......**

Example1.

Calculate the value: $\begin{vmatrix} 1 & 2 & 4 & -1 \\ 2 & 2 & -3 & 5 \\ -3 & 1 & 1 & -2 \\ 4 & 2 & 1 & -2 \end{vmatrix}$

Solution:

The idea is:, using properties of the determinant No.8

$$\begin{vmatrix} 1 & 2 & 4 & -1 \\ 2 & 2 & -3 & 5 \\ -3 & 1 & 1 & -2 \\ 4 & 2 & 1 & -2 \end{vmatrix}$$

First, we make zero where is 4 . **How? I row * (- 4) + IV row** $\longrightarrow$ **goes to IV row**

$1\circ$(-4)+4=0
$2\circ$(-4)+2= -6
$4\circ$(-4)+1= -15
(-1)(-4)+(-2)=2

$$\begin{vmatrix} 1 & 2 & 4 & -1 \\ 2 & 2 & -3 & 5 \\ -3 & 1 & 1 & -2 \\ 4 & 2 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 & -1 \\ 2 & 2 & -3 & 5 \\ -3 & 1 & 1 & -2 \\ 0 & -6 & -15 & 2 \end{vmatrix}$$

Next, we make zero where is - 3.

I row *3 + III row → III row

$1 \circ 3+(-3)=0$
$2 \circ 3+1 = 7$
$4 \circ 3+1 =13$
$(-1) \circ 3+(-2)= -5$

$$\begin{vmatrix} 1 & 2 & 4 & -1 \\ 2 & 2 & -3 & 5 \\ -3 & 1 & 1 & -2 \\ 0 & -6 & -15 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 & -1 \\ 2 & 2 & -3 & 5 \\ 0 & 7 & 13 & -5 \\ 0 & -6 & -15 & 2 \end{vmatrix}$$

And finaly, we make zero where is 2.

(-2)*I row + II row → II row

$(-2) \circ 1+2= 0$
$(-2) \circ 2+2= -2$
$(-2) \circ 4+(-3)=-11$
$(-2)(-1)+5=7$

$$\begin{vmatrix} 1 & 2 & 4 & -1 \\ 2 & 2 & -3 & 5 \\ 0 & 7 & 13 & -5 \\ 0 & -6 & -15 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 & -1 \\ 0 & -2 & -11 & 7 \\ 0 & 7 & 13 & -5 \\ 0 & -6 & -15 & 2 \end{vmatrix}$$

If you develop determined by the first column, we will have only one member!

$$\begin{vmatrix} 1 & 2 & 4 & -1 \\ 0 & -2 & -11 & 7 \\ 0 & 7 & 13 & -5 \\ 0 & -6 & -15 & 2 \end{vmatrix} = \begin{vmatrix} -2 & -11 & 7 \\ 7 & 13 & -5 \\ -6 & -15 & 2 \end{vmatrix}$$ Rule of Sarrus …

$$\begin{vmatrix} -2 & -11 & 7 \\ 7 & 13 & -5 \\ -6 & -15 & 2 \end{vmatrix} \begin{matrix} -2 & -11 \\ 7 & 13 \\ -6 & -15 \end{matrix} = \text{-52-330-735+154+150+546} = -267$$

- - -

+ + +

Of course you do not need to go step by step, but immediately take all three zero!

Example 2. Calculate $\begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix} = ?$

Solution: $\begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix}$ below a in the first column, we have to made all zero!

II row – I row ⟶ **II row**

III row – I row ⟶ **III row**

IV row – I row ⟶ **IV row**

$$\begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix} = \begin{vmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{vmatrix}$$ ⟵ From the first row we can draw a **common** a .

$$\begin{vmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{vmatrix} = a \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{vmatrix}$$ ⟶ develop by 1 column

$$a \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{vmatrix} = a \begin{vmatrix} b-a & b-a & b-a \\ b-a & c-a & c-a \\ b-a & c-a & d-a \end{vmatrix} = a(b-a) \begin{vmatrix} 1 & b-a & b-a \\ 1 & c-a & c-a \\ 1 & c-a & d-a \end{vmatrix}$$

Go again to make the zero in the first column!

- II row – I row $\longrightarrow$ II row : c-a-b+a=c-b

- III row – I row $\longrightarrow$ III row : c-a-b+a=c-b and d-a-b+a=d-b

$$a(b-a)\begin{vmatrix}1 & b-a & b-a\\ 1 & c-a & c-a\\ 1 & c-a & d-a\end{vmatrix}=a(b-a)\begin{vmatrix}1 & b-a & b-a\\ 0 & c-b & c-b\\ 0 & c-b & d-b\end{vmatrix}=a(b-a)\begin{vmatrix}c-b & c-b\\ c-b & d-b\end{vmatrix}=a(b-a)(c-b)(d-c)$$

So, the solution is:

$$\begin{vmatrix}a & a & a & a\\ a & b & b & b\\ a & b & c & c\\ a & b & c & d\end{vmatrix}=a(b-a)(c-b)(d-c)$$